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# The $m_W - m_Z$ interdependence in the Standard Model: a new scrutiny

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**ABSTRACT:** The  $m_W - m_Z$  interdependence in the Standard Model is studied in the  $\overline{MS}$  scheme at the two-loop level, including the known higher-order contributions. The relevant radiative parameters,  $\Delta\hat{\alpha}(\mu)$ ,  $\Delta\hat{r}_W$ ,  $\hat{\rho}$  are computed at  $O(\alpha^2)$  taking into account higher-order QCD corrections and the resummation of the reducible contributions. We obtain  $m_W = 80.357 \pm 0.009 \pm 0.003$  GeV where the errors refer to the parametric and theoretical uncertainties, respectively. A comparison with the known result in the On-Shell scheme gives a difference of  $\approx 6$  MeV. As a byproduct of our calculation we also obtain the  $\overline{MS}$  electromagnetic coupling and the weak mixing angle at the top mass scale,  $\hat{\alpha}(M_t) = (127.73)^{-1} \pm 0.0000003$  and  $\sin^2\hat{\theta}_W(M_t) = 0.23462 \pm 0.00012$ .

**KEYWORDS:** Standard Model, LEP HERA and SLC Physics

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## 1 Introduction

The discovery of a new scalar resonance with mass around 125 GeV and properties compatible with those of the Standard Model (SM) Higgs boson at the Large Hadron Collider (LHC) [1, 2] has completed the search for the particles foreseen in the SM. The first run of the LHC has delivered two important messages: i) no signal of physics beyond the SM (BSM) was observed. ii) The Higgs boson was found exactly in the mass range 110-160 GeV predicted by the SM. This indicates that BSM physics, if it exists, is likely to be at a high scale, and possibly out of the reach of direct LHC searches, in which case BSM physics could be constrained only indirectly using high precision measurements. To this end, the precision of the SM predictions should match the experimental one and the theoretical uncertainties should be reliably estimated.

Among the various precision observables the  $W$  boson mass,  $m_W$ , has always played a very important role. Historically, the inclusion of the radiative corrections in the prediction of  $m_W$  in the SM from the electromagnetic coupling,  $\alpha$ , the Fermi constant,  $G_\mu$ , and the weak mixing angle,  $\theta_W$ , as extracted from deep inelastic neutrino scattering [3], was the main motivation to develop the On-Shell (OS) renormalization scheme [4]. In the OS scheme  $\theta_W$  is defined in terms of the pole masses of the  $W$  and  $Z$  bosons,  $\sin^2\theta_W \equiv s^2 = 1 - m_W^2/m_Z^2$ , and the tree level relation between  $G_\mu$  and  $m_W$  is corrected by the radiative parameter  $\Delta r$  via

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi\alpha}{2m_W^2 s^2} [1 + \Delta r] \quad (1.1)$$

that gives rise to an  $m_W - m_Z$  interdependence expressed by

$$m_W^2 = \frac{m_Z^2}{2} \left\{ 1 + \left[ 1 - \frac{4A^2}{m_Z^2} (1 + \Delta r) \right]^{1/2} \right\}, \quad (1.2)$$

where  $A = (\pi\alpha/(\sqrt{2}G_\mu))^{1/2} = 37.2804(3) \text{ GeV}$ .

Since the pioneering one-loop computation of  $\Delta r$  reported in ref. [4] many studies have been devoted to the calculation of higher-order (two or more loops) effects in  $\Delta r$ . First, the higher-order contribution related to the iteration of the large one-loop term of  $\mathcal{O}(\alpha \ln(m_Z/m_f))$ , where  $m_f$  is a generic light fermion mass was investigated [5]. Then, strong and electroweak (EW) corrections to the one-loop  $\delta\rho$  contribution, and in particular the effects proportional to powers of the top mass, were investigated in detail for vanishing bottom mass. The  $\mathcal{O}(\alpha\alpha_s)$  contribution to  $\delta\rho$  was obtained in ref. [6], and later the three-loop calculation  $\mathcal{O}(\alpha\alpha_s^2)$  was also accomplished [7–9]. Concerning the EW corrections, the leading two-loop contribution  $\mathcal{O}(\alpha^2 M_t^4/m_W^4)$  to  $\delta\rho$  was first obtained in the large top-mass limit [10], neglecting all the other masses including the Higgs mass, and then in the so-called gaugeless limit of the SM, i.e. in the limit  $g, g' \rightarrow 0$  where  $g$  ( $g'$ ) is the SU(2) (U(1)<sub>Y</sub>) gauge coupling [11–13]. The incorporation of these effects in  $\Delta r$  was addressed in ref. [14]. Needless to say, these calculations were instrumental in the successful prediction of the top mass before its actual discovery. The two-loop knowledge of  $\Delta r$  was later improved with the evaluation of the next-to-leading effects in the heavy top expansion, namely the  $\mathcal{O}(\alpha^2 M_t^2/m_W^2)$  contributions [15, 16]. The latter turned out to be comparatively large and allowed for a drastic reduction of the scheme dependence. Leading three and four-loop effects related to  $\delta\rho$ , in particular the  $\mathcal{O}(\alpha^3 M_t^6/m_W^6)$ ,  $\mathcal{O}(\alpha^2\alpha_s M_t^4/m_W^4)$  [17, 18] and the  $\mathcal{O}(\alpha\alpha_s^3 M_t^2/m_W^2)$  [19, 20] contributions, were also investigated.

The complete calculation of  $\Delta r$  at the two-loop level was accomplished in several steps. First, the  $\mathcal{O}(\alpha\alpha_s)$  corrections were obtained from the full QCD corrections to the gauge bosons self-energies [21–24]. Then the two-loop fermionic contribution, i.e. two-loop diagrams with at least one closed fermion loop, was derived [25–27], and finally the purely bosonic contribution was also obtained [28–30], completing the two-loop computation of  $\Delta r$ . The prediction of  $m_W$  from eq. (1.2) at the two-loop accuracy, including also known three-loop effects, was summarized in ref. [31] by a simple formula that parameterizes the result in term of the relevant input quantities, used by several groups in their fits to the EW precision observables [32–35]. Ref. [31] also estimated the uncertainty due to unknown higher-order effects,  $\delta m_W^{th} \approx 4 \text{ MeV}$ , from the size of the computed three-loop corrections.

The present experimental world average  $m_W^{\text{exp}} = 80.385 \pm 0.015 \text{ GeV}$  agrees well with the indirect determination of  $m_W$  via a full fit to EW precision observables (except  $m_W$ ): ref. [34] reports  $m_W^{\text{fit}} = 80.359 \pm 0.011$ , consistent with the result of ref. [35],  $m_W^{\text{fit}} = 80.362 \pm 0.007$ . The difference between  $m_W^{\text{exp}}$  and  $m_W^{\text{fit}}$  is slightly more than one standard deviation. In view of possible future improvements in the experimental accuracy at the LHC, it is therefore worthwhile to reconsider the indirect determination of  $m_W$  and in particular its theoretical uncertainty. In this paper we study the  $m_W - m_Z$  interdependence at the two-loop level following a path different from the one employed so far, i.e. the

two-loop determination of  $\Delta r$  in the OS scheme, and we critically re-examine the overall theoretical uncertainty of the SM prediction of  $m_W$ .

The  $\overline{MS}$  formulation of the radiative corrections in the SM developed in refs. [36–38] provides an alternative way to address the  $m_W - m_Z$  interdependence. In this framework the gauge coupling constants are defined as  $\overline{MS}$  quantities, while all the masses are interpreted as pole quantities.<sup>1</sup> All gauge couplings are then reexpressed in terms of the  $\overline{MS}$  weak mixing angle  $\hat{\theta}_W(\mu)$  and the  $\overline{MS}$  electromagnetic coupling  $\hat{\alpha}(\mu)$ , defined at the 't-Hooft mass scale  $\mu$ , usually chosen to be equal to  $m_Z$ . The important feature of these two  $\overline{MS}$  parameters is that they are constructed to include all reducible contributions, i.e. the iteration of lowest order terms. In particular,  $\hat{\alpha}(m_Z)$  automatically incorporates the  $\mathcal{O}(\alpha^n \ln^n m_Z/m_f)$  contributions, while  $\sin^2 \hat{\theta}_W \equiv \hat{s}^2$  is free of the  $\mathcal{O}((\alpha M_t^2/m_W^2)^n)$  contributions that in the OS scheme are induced by the renormalization of the OS  $\theta_W$  angle. Similarly, the on-shell masses of the vector bosons automatically absorb the non-decoupling contributions of heavy particles to their self-energies. It follows that in this hybrid scheme higher order effects are expected to be better under control with respect to the OS or a pure  $\overline{MS}$  schemes.

In the  $\overline{MS}$  formulation the  $m_W - m_Z$  interdependence is expressed in terms of three parameters  $\Delta \hat{r}_W$ ,  $\Delta \hat{\alpha}$  and  $\hat{\rho}$ , defined by

$$\begin{aligned} \frac{G_\mu}{\sqrt{2}} &= \frac{\pi \hat{\alpha}(m_Z)}{2m_W^2 \hat{s}^2} [1 + \Delta \hat{r}_W], & \hat{\alpha}(m_Z) &= \frac{\alpha}{1 - \Delta \hat{\alpha}(m_Z)}, \\ \hat{\rho} &= \frac{m_W^2}{m_Z^2 \hat{c}^2} = \frac{c^2}{\hat{c}^2} \end{aligned} \quad (1.3)$$

where  $\hat{c}^2 = 1 - \hat{s}^2$ . Eqs. (1.3) allow for an iterative evaluation of  $\hat{s}^2$  from  $m_Z, \alpha, G_\mu$ :

$$\hat{s}^2 = \frac{1}{2} \left\{ 1 - \left[ 1 - \frac{4\hat{A}^2}{m_Z^2 \hat{\rho}} (1 + \Delta \hat{r}_W) \right]^{1/2} \right\}, \quad (1.4)$$

where  $\hat{A} = (\pi \hat{\alpha}(m_Z)/(\sqrt{2} G_\mu))^{1/2}$ . The analogue for  $m_W$  reads

$$m_W^2 = \frac{\hat{\rho} m_Z^2}{2} \left\{ 1 + \left[ 1 - \frac{4\hat{A}^2}{m_Z^2 \hat{\rho}} (1 + \Delta \hat{r}_W) \right]^{1/2} \right\}. \quad (1.5)$$

The present knowledge of  $\hat{\alpha}(m_Z)$ ,  $\Delta \hat{r}_W$ ,  $\hat{\rho}$  can be summarized as follows: a complete EW two-loop calculation for  $\hat{\alpha}(m_Z)$  was presented in ref. [39]. The other two-parameters are not known at the same level of accuracy: they are only known at the second order in the heavy top expansion, i.e. up to the two-loop  $\mathcal{O}(\alpha^2 M_t^2/m_W^2)$  contributions [15, 16]. In this paper we upgrade the  $\overline{MS}$  calculation at the full two-loop level presenting the complete  $\mathcal{O}(\alpha \alpha_s)$  and  $\mathcal{O}(\alpha^2)$  determination of  $\hat{\alpha}(m_Z)$ ,  $\Delta \hat{r}_W$ ,  $\hat{\rho}$  augmented by the known three-loop corrections.

A result for the two-loop bosonic contribution to  $\Delta r$  in the  $\overline{MS}$  scheme was presented in ref. [30], which differs in several respects from the present work. Indeed, ref. [30] considered

<sup>1</sup>We generically refer to this approach as  $\overline{MS}$  scheme, although it is actually a hybrid OS- $\overline{MS}$  scheme.

only the bosonic contribution to the two-loop corrections to  $\Delta r$  in the  $\overline{MS}$  scheme with all couplings and masses identified with minimally subtracted quantities. Instead, we consider here the complete two-loop corrections to the three radiative parameters,  $\hat{\alpha}(m_Z)$ ,  $\Delta\hat{r}_W$ ,  $\hat{\rho}$ , in the hybrid  $\overline{MS}$  framework discussed above, where on-shell masses are employed.

The precise knowledge of  $m_W$  in the  $\overline{MS}$  framework allows us to estimate the uncertainty of the  $m_W$  prediction in two different ways: i) from the scale dependence of our  $\overline{MS}$  result by varying the 't Hooft mass scale in a large interval between 50 and 500 GeV. ii) From the scheme dependence by comparing our result in the  $\overline{MS}$  scheme with the known result in the OS scheme present in the literature.

As a byproduct of our  $\overline{MS}$  calculation, we also obtain the values of the  $\overline{MS}$  gauge couplings at the weak scale with a two-loop precision. The latter can be used as initial conditions for studies of the renormalization group evolution.

The paper is organized as follows: in the next section we outline our computation. Section 3 discusses the two-loop determination of  $\hat{\alpha}(m_Z)$ ,  $\Delta\hat{r}_W$ ,  $\hat{\rho}$ . Section 4 contains our results for  $\hat{\alpha}(\mu)$ ,  $\sin^2\hat{\theta}_W(\mu)$  and  $m_W$ . In the last section we discuss the uncertainty on the theoretical determination of  $m_W$  and present our conclusions.

## 2 Outline of the computation

In this section we first extend at the two-loop level the  $\overline{MS}$  framework developed at one-loop in refs. [36–38]. Then some technical details concerning our computation are outlined.

The parameters that in our computation require a two-loop renormalization are the two gauge couplings,  $g$ ,  $g'$ , and the masses of the gauge bosons. Actually, as the gauge sector of the SM is described by only 3 parameters,  $g$ ,  $g'$  and  $v$ , the vacuum expectation value (vev) of the Higgs field, once the two gauge couplings are defined as  $\overline{MS}$ -subtracted quantities, one needs to define the mass of only one gauge boson, either the  $W$  or the  $Z$ , while the renormalized mass of the other boson is obtained using the bare relation  $m_{Z_0} = m_{W_0}/\cos\theta_{W_0}$ . We first identify our vev as the minimum of the radiatively corrected scalar potential. The latter implies that all tadpole contributions are cancelled by a tadpole counterterm and that tadpole diagrams do not enter in our computation. We choose to define our renormalized  $W$  mass,  $m_W$ , as a pole quantity fixing our third renormalization condition. Our renormalized  $Z$  mass,  $\hat{m}_Z$ , is a derived quantity identified with  $\hat{m}_Z \equiv m_W/\hat{c}$ . The use of the experimental quantity  $m_Z^{\text{exp}}$  as input in eqs. (1.4), (1.5) requires the derivation, at the two-loop level, of the relation between  $\hat{m}_Z$  and  $m_Z^{\text{exp}}$ .

According to our choice of pole mass for the  $W$  boson, at the one-loop level  $m_W$  can be directly identified with  $m_W^{\text{exp}}$  and its the counterterm,  $\delta m_W^2$ , is given by:

$$\delta^{(1)}m_W^2 = \text{Re} A_{WW}^{(1)}(m_W^2) \quad (2.1)$$

where, in general,  $A_{XY}(q^2)$  is the term proportional to  $g^{\mu\nu}$  in the  $XY$  self-energy and the superscript indicates the loop order. Because of our condition on the cancellation of the tadpoles, no tadpole term is included in eq. (2.1)

At the two-loop level the definition of a pole mass for an unstable gauge boson presents some subtlety in its relation with the corresponding experimental quantity. Since the

beginning of the nineties it was noticed [40–42] that, beyond one-loop order, there is a difference between the mass defined as the pole of the real part of the propagator (labelled  $m$ ), or as the real part of the complex pole of the  $S$  matrix,  $M$  in the following. We recall here the discussion on the  $Z$  mass developed in ref. [40–42] that can also be applied to the  $W$  case. The former definition leads to the  $Z$  mass counterterm

$$\delta m_Z^2 = \text{Re } A_{ZZ}(m_Z^2) \quad (2.2)$$

that, at the two-loop level, depends on the gauge parameter,  $\xi$ , if the r.h.s. of eq. (2.2) is evaluated in a gauge with  $\xi < (4 \cos^2 \theta_W)^{-1}$ , while it is independent of  $\xi$  if the evaluation is performed with  $\xi \geq (4 \cos^2 \theta_W)^{-1}$ . Let us now denote by  $\bar{s}$  the position of the complex pole of the  $Z$  propagator. Hence

$$\bar{s} = M_{Z_0}^2 + A_{ZZ}(\bar{s}), \quad (2.3)$$

where  $M_{Z_0}$  is the bare mass. The complex pole definition of the renormalized mass and width of the  $Z$  boson follows immediately,

$$\bar{s} = M_Z^2 - i M_Z \Gamma_Z, \quad (2.4)$$

and gives rise to a two-loop mass counterterm given by

$$\delta^{(2)} M_Z^2 = \text{Re } A_{ZZ}^{(2)}(M_Z^2) + \text{Im } A_{ZZ}'(M_Z^2) M_Z \Gamma_Z. \quad (2.5)$$

The  $Z$  boson mass defined according to the real part of the complex pole of the  $S$  matrix generates a fixed-width Breit-Wigner behavior of the total cross section while  $m_Z^{\text{exp}}$  is extracted using a Breit-Wigner parametrization with an energy dependent width. This introduces a mismatch among the parameters entering the r.h.s. of eq. (2.4) and their experimental counterparts that is corrected by [40–43]:

$$M_Z = m_Z^{\text{exp}} \left[ 1 + \left( \frac{\Gamma_Z^{\text{exp}}}{m_Z^{\text{exp}}} \right)^2 \right]^{-1/2}, \quad \Gamma_Z = \Gamma_Z^{\text{exp}} \left[ 1 + \left( \frac{\Gamma_Z^{\text{exp}}}{m_Z^{\text{exp}}} \right)^2 \right]^{-1/2}. \quad (2.6)$$

On the other hand,  $m_Z$  defined as the pole of the real part of the propagator can be directly identified with  $m_Z^{\text{exp}}$  if one works at the two-loop level evaluating eq. (2.2) in a gauge with  $\xi \geq (4 \cos^2 \theta_W)^{-1}$ .

We decided to identify our renormalized  $W$  mass directly with the quantity extracted experimentally. According to the above discussion this fixes  $\delta^{(2)} m_W^2$  to be

$$\delta^{(2)} m_W^2 = \text{Re } A_{WW}^{(2)}(m_W^2) \quad (2.7)$$

with the understanding that the r.h.s. of eq. (2.7) has to be evaluated in a gauge where spurious gauge-dependent terms do not arise. The same condition applies to the relation between  $\hat{m}_Z$  and  $m_Z$ , which is identified with  $m_Z^{\text{exp}}$ . We fulfill it by evaluating  $\text{Re } A_{WW}^{(2)}(m_W^2)$  and  $\text{Re } A_{ZZ}^{(2)}(m_Z^2)$  in the  $\xi = 1$  Feynman gauge. We stress that with our choice the  $m_W$  prediction of eq. (1.5) can be *directly* compared with  $m_W^{\text{exp}}$ . The other possible definition

of the  $W$  mass,  $M_W$ , requires instead the correction factor of eq. (2.6) before it can be compared with  $m_W^{\text{exp}}$ .

The other mass parameters that enter our computation require only a one-loop definition. We define the Higgs, top and bottom masses as pole quantities. The bottom mass is set different from zero only in the one-loop contribution and in the  $\mathcal{O}(\alpha\alpha_s)$  corrections. All other quarks are taken massless. The leptons are also taken massless except for the evaluation of  $\hat{\alpha}$  where the experimental values in the Particle Data Group [44] have been used.

We conclude this section outlining some technical details concerning our computation. All the diagrams entering the calculation of  $\hat{\alpha}(m_Z)$ ,  $\Delta\hat{r}_W$ ,  $\hat{\rho}$  were generated using the Mathematica package FEYNARTS [45]. The reduction of the two-loop diagrams to scalar integrals was done using the code TARCER [46] which uses the algorithm by Tarasov [47] and is now part of the FEYNALC [48] package. In order to extract the vertex and box contributions in  $\Delta\hat{r}_W$  from the relevant diagrams, we used the projector presented in ref. [28, 29]. After the reduction to scalar integrals we were left with the evaluation of two-loop vacuum integrals and two-loop self-energy diagrams at external momenta different from zero. The former integrals were evaluated analytically using the results of ref. [49]. The latter ones were instead reduced to the set of loop-integral basis functions introduced in ref. [50]. The evaluation of the basis functions was done numerically using the code TSIL [51] that, according to the authors, reaches a relative accuracy better than  $10^{-10}$  in the evaluation of integrals without large hierarchies in the masses.

All our results were obtained in the  $R_\xi$  gauge with  $\xi = 1$  and cross-checked in the  $\xi = 1$  background field method (BFM) gauge. The two-point function of a particle, i.e. the sum of the self-energy and of the tadpole diagrams, when evaluated on-shell represent a physical amplitude and must be gauge-invariant. Enforcing the cancellation of the tadpoles, we verified that the sum of the one-particle-irreducible and counterterms diagrams in  $\text{Re}A_{WW}^{(2)}(m_W^2)$  and  $\text{Re}A_{ZZ}^{(2)}(m_Z^2)$  gives the same result in the two gauges.

### 3 Two-loop determination of $\hat{\alpha}(m_Z)$ , $\Delta\hat{r}_W$ , $\hat{\rho}$

In this section we present the two-loop contributions to the three radiative parameters of the  $\overline{MS}$  scheme. To properly identify the two-loop contribution to these parameters the exact specification of the corresponding one-loop result is needed. In the appendix we report the one-loop expressions for  $\hat{\alpha}(m_Z)$ ,  $\Delta\hat{r}_W$ ,  $\hat{\rho}$  that we employed in our computation.

#### 3.1 $\hat{\alpha}(m_Z)$

The evaluation of the electromagnetic coupling in the  $\overline{MS}$  scheme at the two-loop level was discussed in ref. [39]. Here we just recall the main features of that analysis and update the QCD corrections.

The analysis starts from the observation that in the Feynman BFM gauge the renormalization of the electric charge is given only by self-energy diagrams making manifest the possibility of a Dyson summation. From the relation between the bare and the renormalized

$m_Z = 91.1876 \pm 0.0021 \text{ GeV}$
$m_H = 125.15 \pm 0.24 \text{ GeV}$
$M_t = 173.34 \pm 0.76_{\text{exp}} \pm 0.3_{th} \text{ GeV}$
$m_e = 0.510998928 \pm 0.000000011 \text{ MeV}$
$m_\mu = 105.6583715 \pm 0.0000035 \text{ MeV}$
$m_\tau = 1776.82 \pm 0.16 \text{ MeV}$
$m_b = 4.8 \pm 0.3 \text{ GeV}$
$G_\mu = 1.1663781 \pm 0.0000006 \times 10^{-5} \text{ GeV}^{-2}$
$\alpha_s(m_Z) = 0.1184 \pm 0.0007$
$\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) = 0.02750 \pm 0.00033$

**Table 1.** Experimental input values used in our analysis

electric charge defined at zero momentum transfer

$$e^2 = \frac{e_0^2}{1 - e_0^2 \Pi_{\gamma\gamma}(0)}, \quad (3.1)$$

where  $\Pi_{\gamma\gamma}$  is related to the transverse part of the photon self-energy  $A_{\gamma\gamma}(q^2)$  by

$$A_{\gamma\gamma}(q^2) = q^2 e_0^2 \Pi_{\gamma\gamma}(q^2) \quad (3.2)$$

it is easy to derive the relation between  $\alpha = (137.035999074)^{-1}$  and the electromagnetic coupling in the  $\overline{MS}$  scheme at the scale  $\mu$

$$\hat{\alpha}(\mu) = \frac{\alpha}{1 - \Delta\hat{\alpha}(\mu)} \quad (3.3)$$

with

$$\Delta\hat{\alpha}(\mu) = -4\pi\alpha\Pi_{\gamma\gamma}(0)|_{\overline{MS}} \quad (3.4)$$

where  $\overline{MS}$  is denoting the  $\overline{MS}$  renormalization. As we are interested in the evaluation of  $\hat{\alpha}(\mu)$  in the SM at a scale below  $\mu = M_t$  we do not apply the decoupling of the top contribution from  $\Pi_{\gamma\gamma}(0)$ .

The vacuum polarization function in eq. (3.4) can be organized into the sum of a bosonic and a fermionic contribution, the latter defined as arising from diagrams where both the external photons couple to fermions,

$$\Pi_{\gamma\gamma}(0) = \Pi_{\gamma\gamma}^{(f)}(0) + \Pi_{\gamma\gamma}^{(b)}(0). \quad (3.5)$$

The fermionic contribution can be further split into a leptonic part,  $\Pi_{\gamma\gamma}^{(l)}$ , a perturbative quark contribution,  $\Pi_{\gamma\gamma}^{(p)}$ , and a non-perturbative one,  $\Pi_{\gamma\gamma}^{(5)}(0)$ . The latter, associated to diagrams in which a light quark couples to the external photons with no heavy masses circulating in the loops, can be related to the hadronic contribution to the vacuum polarization  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) \equiv 4\pi\alpha \left( \text{Re} \Pi_{\gamma\gamma}^{(5)}(m_Z^2) - \Pi_{\gamma\gamma}^{(5)}(0) \right)$  so that

$$\begin{aligned} \Pi_{\gamma\gamma}^{(f)}(0) &= \Pi_{\gamma\gamma}^{(l)}(0) + \Pi_{\gamma\gamma}^{(p)}(0) + \Pi_{\gamma\gamma}^{(5)}(0) \\ &= \Pi_{\gamma\gamma}^{(l)}(0) + \Pi_{\gamma\gamma}^{(p)}(0) + \left( \Pi_{\gamma\gamma}^{(5)}(0) - \text{Re} \Pi_{\gamma\gamma}^{(5)}(m_Z^2) \right) + \text{Re} \Pi_{\gamma\gamma}^{(5)}(m_Z^2). \end{aligned} \quad (3.6)$$



The hadronic contribution can be obtained from the experimental data on the cross section in  $e^+e^- \rightarrow \text{hadrons}$  by using a dispersion relation. Two recent evaluations of  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2)$  report very consistent results:  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) = (275.7 \pm 1.0) \times 10^{-4}$  [52],  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) = (275.0 \pm 3.3) \times 10^{-4}$  [53]. We use the latter as reference value in our calculation. The  $\Pi_{\gamma\gamma}^{(p)}$  term in eq. (3.6) includes the top contribution to the vacuum polarization plus the two-loop diagrams in which a light quark couples internally to the  $W$  and  $Z$  bosons. This contribution, as well as  $\text{Re } \Pi_{\gamma\gamma}^{(5)}(m_Z^2)$ , can be safely analyzed perturbatively.

The one-loop contribution to  $\Delta\hat{\alpha}^p(m_Z) \equiv \Delta\hat{\alpha}(m_Z) - \Delta\alpha_{\text{had}}^{(5)}(m_Z^2)$  is reported in eq. (A.3) of the appendix. The higher order contributions to  $\Delta\hat{\alpha}^p(m_Z)$  are presented here as a simple formula that parametrizes the full result in terms of the top and the Higgs masses, the strong coupling, and  $\hat{s}^2$ :

$$\Delta\hat{\alpha}^{p, h.o.}(m_Z) = 10^{-4} (b_0 + b_1 ds + b_2 dT + b_3 dH + b_4 da_s) \quad (3.7)$$

where

$$\begin{aligned} ds &= \left( \frac{\hat{s}^2}{0.231} - 1 \right), & dT &= \ln \left( \frac{M_t}{173.34 \text{ GeV}} \right), \\ dH &= \ln \left( \frac{m_H}{125.15 \text{ GeV}} \right), & da_s &= \left( \frac{\alpha_s(m_Z)}{0.1184} - 1 \right) \end{aligned} \quad (3.8)$$

with

$$b_0 = 1.751181 \quad b_1 = -0.523813, \quad b_2 = -0.662710, \quad b_3 = -0.000962, \quad b_4 = 0.252884. \quad (3.9)$$

Eq. (3.7) includes the  $\mathcal{O}(\alpha)$  contribution<sup>2</sup> to  $\Pi_{\gamma\gamma}^{(b)}(0) + \Pi_{\gamma\gamma}^{(l)}(0) + \Pi_{\gamma\gamma}^{(p)}(0)$  plus the  $\mathcal{O}(\alpha_s)$  corrections to  $\Pi_{\gamma\gamma}^{(p)}(0)$  and the  $\mathcal{O}(\alpha_s, \alpha_s^2)$  corrections to  $\text{Re } \Pi_{\gamma\gamma}^{(5)}(m_Z^2)$  [54]. It approximates the exact result to better than 0.045% for  $\hat{s}^2$  in the interval  $(0.23 - 0.232)$  when the other parameters in eq. (3.7) are varied simultaneously within a  $3\sigma$  interval around their central values, given in table 1.

### 3.2 $\Delta\hat{r}_W$

The radiative parameter  $\Delta\hat{r}_W$  enters the relation between the Fermi constant and the  $W$  mass. We recall that the Fermi constant is defined in terms of the muon lifetime  $\tau_\mu$  as computed in an effective 4-fermion  $V-A$  Fermi theory supplemented by QED interactions:

$$\frac{1}{\tau_\mu} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} F\left(\frac{m_e^2}{m_\mu^2}\right) (1 + \Delta q) \left(1 + \frac{3m_\mu^2}{5m_W^2}\right), \quad (3.10)$$

where  $F(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln \rho = 0.9981295$  (for  $\rho = m_e^2/m_\mu^2$ ) is the phase space factor and  $\Delta q = \Delta q^{(1)} + \Delta q^{(2)} = (-4.234 + 0.036) \times 10^{-3}$  are the QED corrections computed at one [55] and two loops [56]. The calculation of  $\Delta\hat{r}_W$  requires the subtraction of the QED corrections, matching the result in the SM with that in the Fermi theory

<sup>2</sup>We alert the reader that our  $\Pi_{\gamma\gamma}$  is defined with the  $e_0^2$  coupling extracted, see eqs. (3.1), (3.2); therefore the  $\mathcal{O}(\alpha)$  contribution is actually due to two-loop diagrams.

which is renormalizable to all orders in the electromagnetic interaction but to lowest order in  $G_\mu$ . As discussed in detail in ref. [30], the contribution of the Fermi effective theory vanishes in the limit of vanishing fermion masses and external momenta. Therefore the matching is automatically obtained just computing the SM contribution in this limit. We explicitly verified that the contribution of the Fermi effective theory vanishes when the fermion masses and external momenta are set equal to zero.

The muon-decay amplitude at the two-loop level can be written as

$$\frac{G_\mu}{\sqrt{2}} = \frac{g_0^2}{8m_{W_0}^2} \left\{ 1 - \frac{A_{WW}}{m_{W_0}^2} + V_W + m_{W_0}^2 B_W + \left( \frac{A_{WW}}{m_{W_0}^2} \right)^2 - \frac{A_{WW} V_W}{m_{W_0}^2} \right\} \quad (3.11)$$

where  $g_0$  is the unrenormalized SU(2) coupling,  $m_{W_0}$  is the unrenormalized  $W$  mass,  $A_{WW} \equiv A_{WW}(0)$ , and  $V_W$  and  $B_W$  are the relevant vertex and box contributions to  $\mu$ -decay. Performing the shift  $m_{W_0}^2 \rightarrow m_W^2 - \delta m_W^2$ , and working at the two-loop order we arrive at

$$\begin{aligned} \frac{G_\mu}{\sqrt{2}} = \frac{g_0^2}{8m_W^2} & \left[ 1 + \frac{\delta^{(1)} m_W^2}{m_W^2} - \frac{A_{WW}^{(1)}}{m_W^2} + E^{(1)} + \frac{\delta^{(2)} m_W^2}{m_W^2} - \frac{A_{WW}^{(2)}}{m_W^2} + E^{(2)} \right. \\ & \left. + A_{WW}^{(1)} B_W^{(1)} + \left( \frac{\delta^{(1)} m_W^2}{m_W^2} - \frac{A_{WW}^{(1)}}{m_W^2} \right) \left( \frac{\delta^{(1)} m_W^2}{m_W^2} - \frac{A_{WW}^{(1)}}{m_W^2} + E^{(1)} \right) \right] \end{aligned} \quad (3.12)$$

where the superscript indicated the loop order and  $E^{(i)} \equiv V_W^{(i)} + m_{W_0}^2 B_W^{(i)}$ . Performing an  $\overline{MS}$  renormalization of the SU(2) and U(1) couplings we write

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi \hat{\alpha}(m_Z)}{2m_W^2 \hat{s}^2} [1 + \Delta \hat{r}_W] \quad (3.13)$$

with

$$\Delta \hat{r}_W = \Delta \hat{r}_W^{(1)} + \Delta \hat{r}_W^{(2)}, \quad (3.14)$$

$$\Delta \hat{r}_W^{(1)} = \frac{\text{Re} A_{WW}^{(1)}(m_W^2)}{m_W^2} - \frac{A_{WW}^{(1)}}{m_W^2} + E^{(1)} \Big|_{\overline{MS}}, \quad (3.15)$$

$$\begin{aligned} \Delta \hat{r}_W^{(2)} = & \frac{\text{Re} A_{WW}^{(2)}(m_W^2)}{m_W^2} - \frac{A_{WW}^{(2)}}{m_W^2} + E^{(2)} + \delta^\epsilon \Delta \hat{r}_W^{(1)} + A_{WW}^{(1)} B_W^{(1)} \\ & + \left( \frac{\text{Re} A_{WW}^{(1)}(m_W^2)}{m_W^2} - \frac{A_{WW}^{(1)}}{m_W^2} \right) \left( \frac{\text{Re} A_{WW}^{(1)}(m_W^2)}{m_W^2} - \frac{A_{WW}^{(1)}}{m_W^2} + E^{(1)} \right) \Big|_{\overline{MS}}. \end{aligned} \quad (3.16)$$

where  $\overline{MS}$  in this case denotes both the  $\overline{MS}$  renormalization and the choice  $\mu = m_Z$  for the 't Hooft mass scale;  $\delta^\epsilon \Delta \hat{r}_W^{(1)}$  is the finite contribution proportional to the coefficient of the  $\mathcal{O}(\epsilon)$  part of the one-loop contribution ( $\epsilon = (4-d)/2$  where  $d$  is the dimension of the space-time) induced by the renormalization of  $g_0^2$  in eq. (3.12).

We note that the definition of  $\Delta \hat{r}_W$  in eq. (3.13) differs from the original proposal in ref. [37],

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi \alpha}{2m_W^2 \hat{s}^2} \frac{1}{1 - \Delta \hat{r}_W}. \quad (3.17)$$

In eq. (3.17) the relation between  $G_\mu$  and  $m_W$  is expressed in terms of  $\alpha$  and the mass singularity corrections are directly included in  $\Delta\hat{r}_W$  and their resummation is achieved via the replacement

$$1 + \Delta\hat{r}_W \rightarrow \frac{1}{1 - \Delta\hat{r}_W} . \quad (3.18)$$

This is clearly different from eq. (3.13) where they are absorbed in  $\hat{\alpha}(m_Z)$ . The replacement (3.18) used in eq. (3.17) introduces spurious two-loop and higher-order contributions, numerically quite small. The use of eq. (3.13) allows us instead to control directly the resummation of the various contributions.

An explicit expression for  $\Delta\hat{r}_W^{(1)}$  is reported in eq. (A.3) of the appendix where the distinction between  $\hat{c}^2$  and  $c^2$  is kept. The higher order contributions to  $\Delta\hat{r}_W$  are presented again in a simple formula that approximates the exact result to better than 0.035% for  $\hat{s}^2$  on the interval  $(0.23 - 0.232)$  when the other parameters are varied simultaneously within a  $3\sigma$  interval around their central values. We find

$$\Delta\hat{r}_W^{h.o.}(m_Z) = 10^{-4} (r_0 + r_1 ds + r_2 dT + r_3 dH + r_4 da_s) \quad (3.19)$$

with

$$r_0 = -2.8472779, \quad r_1 = 1.620742, \quad r_2 = 1.773226, \quad r_3 = -0.364310, \quad r_4 = 1.137797 . \quad (3.20)$$

Eq. (3.19) includes, besides the  $\Delta\hat{r}_W^{(2)}$  contribution from eq. (3.16), the complete  $\mathcal{O}(\alpha\alpha_s)$  corrections and the first two subleading terms in the heavy top expansion of the three-loop  $\mathcal{O}(\alpha\alpha_s^2)$  corrections.

### 3.3 $\hat{\rho}$

The relation between the weak mixing angle in the  $\overline{MS}$  formulation and its OS counterpart is encoded in the parameter  $\hat{\rho}$  defined as

$$\hat{\rho} = \frac{c^2}{\hat{c}^2} = \frac{m_W^2}{m_Z^2 \hat{c}^2} \quad (3.21)$$

whose tree-level value is equal to 1. From the relation

$$\frac{m_{W_0}}{m_{Z_0}} \equiv c_0^2 = c^2 - c^2 \frac{\delta m_W^2}{m_W^2} + c_0^2 \frac{\delta m_Z^2}{m_Z^2} = \hat{c}^2 - \delta\hat{c}^2 \quad (3.22)$$

with  $\delta m_Z$  given by eq. (2.2) and  $\delta\hat{c}^2$  the counterterms for  $\hat{c}^2$ , it is easy to derive

$$\hat{\rho} = \frac{1}{(1 - Y_{\overline{MS}})} . \quad (3.23)$$

with

$$Y = \frac{\delta m_W^2}{m_W^2} - c_0^2 \frac{\delta m_Z^2}{m_Z^2} . \quad (3.24)$$

In eq. (3.23)  $\overline{MS}$  denotes both the  $\overline{MS}$  renormalization and the choice  $\mu = m_Z$  for the 't Hooft mass scale. Indeed the structure of the  $1/\epsilon$  poles in  $\delta\hat{c}^2$  is identical to that of the

combination of the  $W$  and  $Z$  mass counterterms in eq. (3.22) once the  $1/\epsilon$  poles in  $\delta^{(1)}m_W^2$  and  $\delta^{(1)}m_Z^2$  are expressed in terms of  $\overline{MS}$  quantities.

The two-loop counterterm  $\delta^{(2)}m_Z^2$  includes also the contribution from the mixed  $\gamma Z$  self-energy or

$$\delta^{(2)}m_Z^2 = \text{Re} \left[ A_{ZZ}^{(1)}(m_Z^2) + A_{ZZ}^{(2)}(m_Z^2) + \left( \frac{A_{\gamma Z}^{(1)}(m_Z^2)}{m_Z^2} \right)^2 \right] \quad (3.25)$$

so that  $Y_{\overline{MS}}$  up to the two-loop level reads

$$Y_{\overline{MS}} = Y_{\overline{MS}}^{(1)} + Y_{\overline{MS}}^{(2)}, \quad (3.26)$$

$$Y_{\overline{MS}}^{(1)} = \text{Re} \left[ \frac{A_{WW}^{(1)}(m_W^2)}{m_W^2} - \hat{c}^2 \frac{A_{ZZ}^{(1)}(m_Z^2)}{m_W^2} \right]_{\overline{MS}}, \quad (3.27)$$

$$Y_{\overline{MS}}^{(2)} = \text{Re} \left[ \frac{A_{WW}^{(2)}(m_W^2)}{m_W^2} - \frac{A_{ZZ}^{(2)}(m_Z^2)}{m_Z^2} + \left( \frac{A_{\gamma Z}^{(1)}}{m_Z^2} \right)^2 \right]_{\overline{MS}}. \quad (3.28)$$

The one-loop contribution to  $Y_{\overline{MS}}$  is reported in eq. (A.4) of the appendix. As before we give the higher order terms via a simple formula:

$$Y_{\overline{MS}}^{h.o.}(m_Z) = 10^{-4} (y_0 + y_1 ds + y_2 dt + y_3 dH + y_4 da_s) \quad (3.29)$$

where  $dt = [(M_t/173.34 \text{ GeV})^2 - 1]$  and

$$y_0 = -18.616753 \quad y_1 = 15.972019, \quad y_2 = -16.216781, \quad y_3 = 0.0152367, \quad y_4 = -13.633472. \quad (3.30)$$

Eq. (3.29) includes, besides the  $Y_{\overline{MS}}^{(2)}$  contribution from eq. (3.28), the complete  $\mathcal{O}(\hat{\alpha}\alpha_s)$  corrections, the leading three-loop  $\mathcal{O}(\hat{\alpha}\alpha_s^2 M_t^2/m_W^2)$  contribution [7, 8] and the subleading  $\mathcal{O}(\hat{\alpha}^3 M_t^6/m_W^6)$  and  $\mathcal{O}(\hat{\alpha}^2 \alpha_s M_t^4/m_W^4)$  [17, 18], and the four-loop  $\mathcal{O}(\hat{\alpha}\alpha_s^3 M_t^2/m_W^2)$  contribution [19, 20]. It approximates the exact result to better than 0.075% for  $\hat{s}^2$  on the interval  $(0.23 - 0.232)$  when the other parameters in eq. (3.29) are varied simultaneously within a  $3\sigma$  interval around their central values.

## 4 Results

In this section we report our results for  $\hat{\alpha}$ ,  $\sin^2 \hat{\theta}_W$  and  $m_W$ . All results are presented as simple parameterizations in terms of the relevant quantities whose stated validity refers to a simultaneous variation of the various parameters within a  $3\sigma$  interval around their central values given in table 1. As a general strategy for the evaluation of the two-loop contributions, where  $\hat{c}^2$  can be identified with  $c^2$ , we have replaced in all the two-loop terms  $m_W$  with  $m_Z \hat{c}$ . This choice gives rise to the weakest  $\mu$ -dependence in  $m_W$ .

The two-loop computation of the  $\overline{MS}$  electromagnetic coupling from eq. (3.3) and of  $\sin^2 \hat{\theta}_W$  from eq. (1.4) can be summarized by the following parameterizations

$$\hat{\alpha}(\mu) = a_0 + 10^{-3} \left( a_1 dH + a_2 dT + a_3 da_s + a_4 da^{(5)} \right) \quad (4.1)$$

$$\sin^2 \hat{\theta}_W(\mu) = s_0 + s_1 dH + s_2 dt + s_3 dH dt + s_4 da_s + s_5 da^{(5)} \quad (4.2)$$

	$\mu = m_Z$	$\mu = M_t$
$a_0$	$(128.13385)^{-1}$	$(127.73289)^{-1}$
$a_1$	-0.00005246	-0.00005267
$a_2$	-0.01688835	0.02087428
$a_3$	0.00014109	0.00168550
$a_4$	0.22909789	0.23057967

	$\mu = m_Z$	$\mu = M_t$
$s_0$	0.2314483	0.2346176
$s_1$	0.0005001	0.0005016
$s_2$	-0.0026004	-0.0001361
$s_3$	0.0000279	0.0000514
$s_4$	0.0005015	0.0004686
$s_5$	0.0097431	0.0098710

**Table 2.** Coefficients for the parameterization of  $\hat{\alpha}(\mu)$  (left table, eq. (4.1) in the text) and  $\sin^2\hat{\theta}_W(\mu)$  (right table, eq. (4.2) in the text).

where  $da^{(5)} = [\Delta\alpha_{\text{had}}^{(5)}(m_Z^2)/0.02750 - 1]$  and the  $a_i$  and  $s_i$  coefficients are reported in table 2 for two different values of the scale  $\mu$ . Eq. (4.1) approximates the exact result to better than  $1.1 \times 10^{-7}$  ( $1.2 \times 10^{-7}$ ) for  $\mu = m_Z$  ( $\mu = M_t$ ), while eq. (4.2) approximates the exact result to better than  $5.1 \times 10^{-6}$  ( $6.2 \times 10^{-6}$ ) for  $\mu = m_Z$  ( $\mu = M_t$ ).

From our results on  $\hat{\alpha}$  and  $\hat{s}^2$  it is easy to obtain the values of the  $g$  and  $g'$  coupling constants at the weak scale, usually identified with  $M_t$ . They can be taken as starting points in the study of the evolution of the gauge couplings via Renormalization Group Equations (RGE) in Grand Unified Models and in the analysis of the stability of the Higgs potential in the SM. Ref. [57] reports the values of the gauge coupling constants at the  $\mu = M_t$  scale,  $g(M_t) = 0.64822$  and  $g'(M_t) = 0.35760$ , obtained using a complete calculation of the two-loop threshold corrections in the SM. Here we find  $g(M_t) = 0.647550 \pm 0.000050$  and  $g'(M_t) = 0.358521 \pm 0.000091$ . The difference between the two results, which should be a three-loop effect, is more sizable than expected. However, the results of ref. [57] were obtained using as input parameters  $G_\mu$  and the experimental values of  $m_Z$  and  $m_W$ , while our result is obtained with a different set of input parameters, i.e.  $G_\mu$ ,  $\alpha$  and  $m_Z$ . In our calculation  $m_W$  is a derived quantity calculable from eq. (1.5). Moreover, as shown below, our prediction for  $m_W$  is not in perfect agreement with the present experimental determination and therefore the gauge couplings extracted using the two different sets of inputs parameters show some discrepancy. Indeed, using our prediction for  $m_W$  in the results of ref. [57] instead of the experimental result, we find that the difference between the  $g$  ( $g'$ ) computed in the two methods is one order of magnitude smaller than the two-loops correction and two orders smaller than the one-loop correction to  $g$  ( $g'$ ).

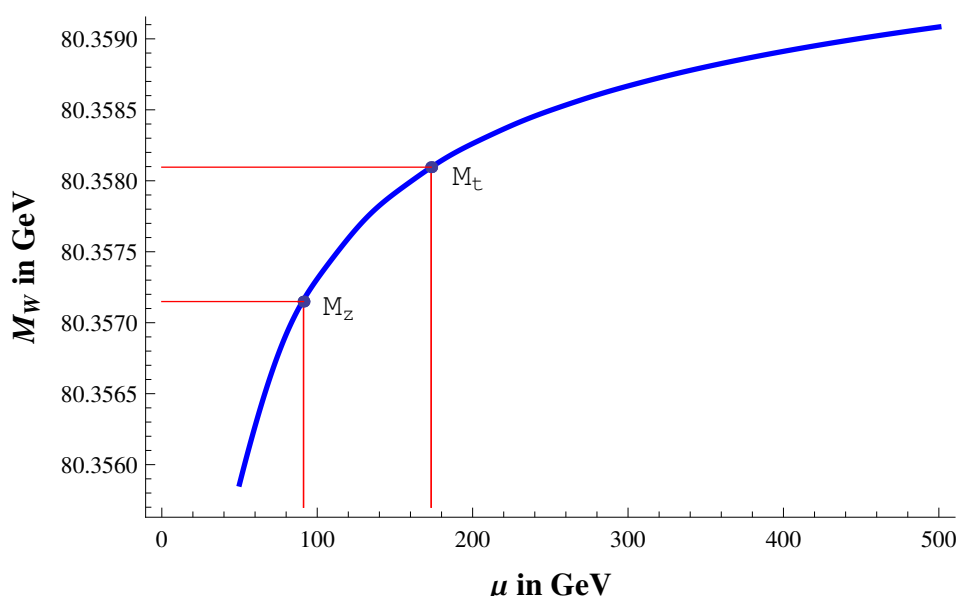
The two-loop determination of the  $W$  mass in the  $\overline{MS}$  framework from eq. (1.5) can be parameterized as follows

$$m_W = w_0 + w_1 dH + w_2 dH^2 + w_3 dh + w_4 dt + w_5 dHdt + w_6 da_s + w_7 da^{(5)} \quad (4.3)$$

with  $dh = [(m_H/125.15 \text{ GeV})^2 - 1]$ . The  $w_i$  coefficients are reported in table 3 for  $\mu = m_Z$ . Two different cases are considered. In the left column the coefficients refer to the standard case of a simultaneous variation of all parameters within a  $3\sigma$  interval around their central values. The right column applies to the case where all parameters but the Higgs mass are varied within a  $3\sigma$  interval while the latter is varied between 50 and 450 GeV. In the

	$124.42 \leq m_H \leq 125.87 \text{ GeV}$	$50 \leq m_H \leq 450 \text{ GeV}$
$w_0$	80.35712	80.35714
$w_1$	-0.06017	-0.06094
$w_2$	0.0	-0.00971
$w_3$	0.0	0.00028
$w_4$	0.52749	0.52655
$w_5$	-0.00613	-0.00646
$w_6$	-0.08178	-0.08199
$w_7$	-0.50530	-0.50259

**Table 3.** Coefficients of the  $m_W$  parameterization in eq. (4.3). The left column contains the coefficients that cover a variation of  $m_H$  around its central value, while the right one applies to the case  $50 \leq m_H \leq 450 \text{ GeV}$ .



**Figure 1.** Dependence of the  $m_W$  prediction on the electroweak scale  $\mu$  in the  $\overline{MS}$  framework.

two cases the formula (4.3) approximates the exact result to better than 0.11 MeV and 0.5 MeV, respectively.

The result for the W mass described by eq. (4.3) is obtained fixing  $\mu = m_Z$ . As a physical quantity, the W mass must be  $\mu$ -independent. Hence the numerical difference between results obtained varying  $\mu$  in a “reasonable” interval can be taken as an indication of the size of the missing higher-order corrections. In figure 1 we plot  $m_W$  vs.  $\mu$ , with the ’t-Hooft mass varying between 50 GeV and 500 GeV. The figure is obtained using as input parameters the central values in table 1. The figure shows a maximum variation of  $\sim 3 \text{ MeV}$  in the entire range while in the restricted range  $100 \leq \mu \leq 200 \text{ GeV}$  we find a maximum variation of  $\sim 1 \text{ MeV}$ .

## 5 Discussion and conclusions

In this paper we have discussed the  $m_W - m_Z$  interdependence in the SM, in the  $\overline{MS}$  framework of the radiative corrections. We have evaluated the parameters  $\hat{\alpha}$ ,  $\Delta\hat{r}_W$  and  $\hat{\rho}$  at the full two-loop level augmented by all the presently known three-loop strong, EW and mixed contributions and by the four-loop strong corrections. We have presented our results via simple formulas that parameterizes the results in terms of  $m_H$ ,  $M_t$ ,  $\alpha_s$  and the 5-flavor hadronic contribution to the vacuum polarization.

Our calculation of the  $W$  mass in the  $\overline{MS}$  framework automatically incorporates the Dyson resummation of the lowest order large contributions, i.e. the mass singularity logarithms and the effects that scale as powers of the top mass. This partial inclusion of terms that are beyond the presently computed effects in the loop expansion is a solid ground to estimate in a realistic way the size of the missing higher-order contributions in the  $m_W$  computation. The very weak residual  $\mu$ -dependence shown in figure 1 indicates that the uncertainty that can be assigned to our  $\overline{MS}$  result due to the truncation of the perturbative series is expected to be at most  $\sim 3$  MeV.

For what concerns the parametric uncertainties, after the discovery of the Higgs boson and the precise measurement of its mass, the most important experimental errors that affects the theoretical determination of  $m_W$  are the ones on  $M_t$  and  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2)$ . The left column of table 3 shows that the sensitivity of  $m_W$  to  $M_t$  is more than twice that to  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2)$ . In our calculation the top mass is an on-shell quantity, i.e. a pole mass, and in table 1 we have identified it with the average of the Tevatron, CMS and ATLAS measurements. However, at the present level of precision of the experimental determination ( $\pm 0.76$  GeV) this identification can be disputed in two aspects. *i)* The top pole mass has an intrinsic non-perturbative ambiguity of  $\mathcal{O}(\Lambda_{QCD})$  due to infrared renormalon effects. *ii)* The top mass parameter extracted by the experiments, which we call  $M_t^{MC}$ , is obtained from the comparison between the kinematical reconstruction of the top quark decay products and the Monte Carlo simulations of the corresponding event. Therefore  $M_t^{MC}$  is a parameter sensitive to the on-shell region of the top quark but it cannot be directly identified with  $M_t$ . The offset between  $M_t$  and  $M_t^{MC}$  is difficult to quantify, and has recently been estimated of  $\mathcal{O}(0.3 - 0.5)$  GeV [58, 59]. In our numerics we have assigned a 1 GeV uncertainty to  $M_t$ .

The  $m_W$  result obtained using the central values in table 1,  $m_W = 80.357$  GeV, agrees within one and a half standard deviations with the present experimental world average,  $m_W = (80.385 \pm 0.015)$  GeV. However, increasing the top mass and decreasing  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2)$  by  $1\sigma$ , i.e. using  $M_t = 174.34$  GeV and  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) = 0.02717$ , we find  $m_W = 80.370$  GeV which is much closer to the experimental world average. It is interesting to note that the precise determination of the top mass plays a very important role also in the analysis of the stability of the SM Higgs potential up to the Planck scale. In order to get a closer agreement between the computed  $m_W$  and the experimental result we saw that large values of  $M_t$  are favored, while vacuum stability in the SM requires quite low values for the top mass,  $M_t < 171.36 \pm 0.46$  GeV [57, 60]. Using  $M_t = 171.36$  GeV and for the other inputs the central values in table 1 we find  $m_W = 80.345$  GeV, which differs from the experimental world average by more than two standard deviations.



Our  $\overline{MS}$  result for  $m_W$  can be compared with the prediction of  $m_W$  in the OS scheme of ref. [31] to study the scheme dependence of the  $m_W$  predictions. Both calculations include the complete two-loop electroweak contributions, higher-order QCD corrections of  $\mathcal{O}(\alpha\alpha_s)$  and  $\mathcal{O}(\alpha\alpha_s^2)$ , the higher-order mixed EW-QCD corrections  $\mathcal{O}(\alpha^2\alpha_s M_t^4)$ , and purely EW  $\mathcal{O}(\alpha^3 M_t^6)$  corrections. In our result also the four-loop contribution  $\mathcal{O}(\alpha\alpha_s^3 M_t^2)$  is included, but we do not take it into account in the comparison with ref. [31]. The  $\overline{MS}$  and OS calculations differ however in several aspects. While in the  $\overline{MS}$  framework we exploit the possibility of resumming lowest-order contributions, no resummation is attempted in the OS calculation. Furthermore, our computation refers directly to  $m_W$ , while in the calculation of ref. [31] the quantity predicted is  $M_W$  (see section 2), which is then translated to  $m_W$  with the introduction of a correction factor containing the  $W$  boson width. Because the latter is not very well known (the experimental uncertainty is presently around 2%), the theoretical result for  $\Gamma_W$  is employed, thus introducing an additional uncertainty in the OS result estimated to be 1-2 MeV [25, 26].

Since the  $m_W$  determinations in the  $\overline{MS}$  and OS scheme are equivalent at the two-loop level but differ by the partial inclusion of higher-order contributions, their numerical difference can be taken as a good estimate of missing higher-order effects. Taking as inputs in our calculation those used in [31], i.e.  $m_H = 100$  GeV,  $M_t = 174.3$  GeV,  $\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) = 0.027572$  and  $\alpha_s = 0.119$ , we find  $m_W = 80.3749$  GeV, which should be compared with the value  $m_W = 80.3800$  reported in ref. [31]. If instead we take the central values in table 1 as inputs in eq. (9) of ref. [31], we find an OS result  $m_W = 80.3639$  GeV to be compared with an  $\overline{MS}$  result  $m_W = 80.3578$  GeV. These numbers indicate that the  $\overline{MS}$  determination is always lower than the OS one, and shows a larger difference with the present experimental world average. Furthermore, the estimate  $\delta m_W^{th} \approx 4$  MeV of the theoretical uncertainty from unknown higher-order corrections reported in ref. [31] seems to be slightly optimistic. A more realistic value is probably  $\delta m_W^{th} \approx 6$  MeV.

Another indication that  $\delta m_W^{th} \approx 4$  MeV is probably an underestimate comes from our  $\overline{MS}$  calculation. In our hybrid scheme the masses that appear in the one-loop contributions are identified with pole masses, and the gauge couplings with  $\overline{MS}$  quantities. In this framework once the one-loop contributions are written as we did in the appendix, the expressions of the two-loop corrections follow. However, their evaluation has some residual ambiguity, because one can always re-express the  $W$  mass as  $m_Z$  and  $\hat{c}$ , or vice versa. As we said, our choice to express  $m_W$  in terms of  $m_Z$  in the two-loop contributions is the one that minimizes the  $\mu$ -dependence, but other choices are allowed. Trying several possibilities, we found a variation of  $\delta m_W \approx 4$  MeV in our  $\overline{MS}$  result;  $m_W$  can be almost 3 MeV below our default choice, further amplifying the difference between the  $\overline{MS}$  and OS schemes.

We have seen that in our  $\overline{MS}$  calculation  $\delta m_W^{th} \approx 3$  MeV, while the scheme dependence observed in the comparison with the OS scheme is around 6 MeV. However, in our  $\overline{MS}$  computation we exploit at best all the present available information through the automatic resummation of the known contributions. Moreover, we expect to have better control over the unknown higher-order contributions than in the OS scheme because in  $\overline{MS}$  the effects related to  $\delta\rho$  are not enhanced by the numerical factor  $c^2/s^2$ . Finally, we predict directly  $m_W$  and not  $M_W$  in order to avoid correction factors that introduce additional uncertainties.



It is therefore natural that the theoretical uncertainty estimated in the  $\overline{MS}$  calculation is smaller than the scheme dependence in the comparison between the OS and  $\overline{MS}$  results.

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## A Appendix

Here we give the explicit formulae for  $\Delta\hat{\alpha}$ ,  $\Delta\hat{r}_W$  and  $Y_{\overline{MS}}$  at the one loop order. In the formulae below  $N_c = 3$  is the number of colors and

$$\zeta_w = \frac{m_H^2}{m_W^2}, \quad \zeta_z = \frac{m_H^2}{m_Z^2}, \quad b_w = \frac{m_b^2}{m_W^2}, \quad b_z = \frac{m_b^2}{m_Z^2}, \quad t_w = \frac{M_t^2}{m_W^2}, \quad t_z = \frac{M_t^2}{m_Z^2}. \quad (\text{A.1})$$

$$\Delta\hat{\alpha}^{p,(1)} = -\frac{\alpha}{4\pi} \left\{ \frac{2}{3} + \frac{4}{3} (\overline{\ln}(m_e^2) + \overline{\ln}(m_\mu^2) + \overline{\ln}(m_\tau^2)) - 7\overline{\ln}(m_W^2) + \frac{16}{27} N_c \overline{\ln}(M_t^2) \right. \\ \left. + N_c \left( -\frac{196}{81} - \frac{4}{27} (1 + 2b_z) B_0(m_Z^2, m_b^2, m_b^2) - \frac{8}{27} b_z \overline{\ln}(m_b^2) + \frac{40}{27} \overline{\ln}(m_Z^2) \right) \right\} \quad (\text{A.2})$$

$$\Delta\hat{r}_W^{(1)} = \frac{\hat{\alpha}}{4\pi\hat{s}^2} \left\{ \frac{1 + 8\hat{c}^2}{12c^4} - \frac{7 + 80\hat{c}^2}{24c^2} + \frac{1}{\hat{c}^2} - 8\hat{c}^2 + \frac{1}{72} (794 - 21\zeta_w + 6\zeta_w^2) \right. \\ + \left( \frac{\hat{c}^2 + 8\hat{c}^4 + 64c^2\hat{c}^4 - 6c^4(8\hat{c}^4 - 5)}{12c^2\hat{c}^2s^2} + \frac{26c^2 - \zeta_w^2 - 18(5 + \zeta_z) + \zeta_w(82 + \zeta_z)}{12s^2(1 - \zeta_w)} \right) \overline{\ln}(m_W^2) \\ + \left( \frac{36c^6 + 24c^4 + 4c^2 - 1}{12c^4s^2} - \frac{3c^2 + 2}{2\hat{c}^2s^2} - \frac{\hat{c}^2(15c^4 - 11c^2 + 2)}{3c^4s^2} \right) \overline{\ln}(m_Z^2) \\ + \frac{\zeta_w(12 - 4\zeta_w + \zeta_w^2)}{12(1 - \zeta_w)} \overline{\ln}(m_H^2) - \frac{12 - 4\zeta_w + \zeta_w^2}{12} B_0(m_W^2, m_H^2, m_W^2) \\ + \left( \frac{11\hat{c}^2 + 1}{3c^2} + 4\hat{c}^2 - \frac{8\hat{c}^2 + 1}{12c^4} - \frac{1}{\hat{c}^2} + 2 \right) B_0(m_W^2, m_Z^2, m_W^2) \\ + N_c \left[ \frac{1}{12} (4b_w t_w - 2b_w^2 - 3b_w - 2t_w^2 - 3t_w - 12) + \frac{2}{3} \overline{\ln}(m_W^2) \right. \\ + \frac{b_w((b_w - t_w)^2 + b_w + 2t_w)}{6(b_w - t_w)} \overline{\ln}(m_b^2) + \frac{t_w((t_w - b_w)^2 + t_w + 2b_w)}{6(t_w - b_w)} \overline{\ln}(M_t^2) \\ \left. + \frac{1}{6} ((t_w - b_w)^2 + t_w + b_w - 2) B_0(m_W^2, M_t^2, m_b^2) \right] \Bigg\} \quad (\text{A.3})$$

$$\begin{aligned}
 Y_{\overline{MS}}^{(1)} = & \frac{\hat{\alpha}}{4\pi\hat{s}^2} \left\{ \frac{1+8\hat{c}^2}{12\hat{c}^4} + \frac{175-416\hat{c}^2+240\hat{c}^4}{36\hat{c}^2} + \frac{262-288\hat{c}^2+3\zeta_W^2-3\zeta_W\zeta_Z}{36} \right. \\
 & + \left( \frac{1+8\hat{c}^2}{12\hat{c}^2} + \frac{\zeta_W-30+64\hat{c}^2-48\hat{c}^4}{12} \right) \overline{\ln}(m_W^2) \\
 & - \left( \frac{1+8\hat{c}^2}{12\hat{c}^4} + \frac{\zeta_Z+34-96\hat{c}^2+48\hat{c}^4}{12\hat{c}^2} \right) \overline{\ln}(m_Z^2) - \frac{1}{12}\zeta_W^2 s^2 \overline{\ln}(m_H^2) \\
 & + \left( \frac{11\hat{c}^2+1}{3\hat{c}^2} + 4\hat{c}^2 - \frac{8\hat{c}^2+1}{12\hat{c}^4} - \frac{1}{\hat{c}^2} + 2 \right) B_0(m_W^2, m_Z^2, m_W^2) \\
 & + \left( \frac{1-4\hat{c}^2-36\hat{c}^4}{12\hat{c}^2} + \frac{5-8\hat{c}^2-12\hat{c}^4}{3} \right) B_0(m_Z^2, m_W^2, m_W^2) \\
 & - \left( \frac{(\zeta_W-4)\zeta_W}{12} + 1 \right) B_0(m_W^2, m_H^2, m_W^2) + \left( \frac{(\zeta_Z-4)\zeta_W}{12} + \frac{1}{\hat{c}^2} \right) B_0(m_Z^2, m_H^2, m_Z^2) \\
 & + N_c \left[ \frac{11-22\hat{c}^2+20\hat{c}^4}{9\hat{c}^2} - \frac{(t_W-b_W)^2}{6} - 1 + \frac{2}{3}\overline{\ln}(m_W^2) - \frac{40\hat{c}^4-44\hat{c}^2+22}{27\hat{c}^2}\overline{\ln}(m_Z^2) \right. \\
 & + \frac{9(b_W-t_W)-8-8\hat{c}^2+16\hat{c}^4}{54} b_W \overline{\ln}(m_b^2) \\
 & + \frac{9(t_W-b_W)+16-80\hat{c}^2+64\hat{c}^4}{54} t_W \overline{\ln}(M_t^2) \\
 & + \frac{5-4\hat{c}^2+8\hat{c}^4+(16\hat{c}^4-8\hat{c}^2-17)b_Z}{54\hat{c}^2} B_0(m_Z^2, m_b^2, m_b^2) \\
 & + \frac{17-40\hat{c}^2+32\hat{c}^4+(64\hat{c}^4-80\hat{c}^2+7)t_Z}{54\hat{c}^2} B_0(m_Z^2, M_t^2, M_t^2) \\
 & \left. + \frac{(t_W-b_W)^2+b_W+t_W-2}{6} B_0(m_W^2, M_t^2, m_b^2) \right\} \quad (A.4)
 \end{aligned}$$

where  $B_0$  is the finite part of the Passarino-Veltman function defined as

$$B_0(s, x, y) = - \int_0^1 dt \overline{\ln}[tx + (1-t)y - t(1-t)s] \quad (A.5)$$

where  $\overline{\ln}(x) = \log\left(\frac{x}{\mu}\right)$  with  $\mu$  the 't Hooft mass.

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